CHAPTER 5

TORSION OF NON-CIRCULAR AND THIN-WALLED SECTIONS

5.1. Rectangular sections

For rectangular shafts, however, with longer side d and shorter side b, it can be shown by experiment that the maximum shearing stress occurs at the centre of the longer side and is given by

$$\tau_{max} = \frac{T}{k_1 d b^2} \qquad \text{...(1)}$$

The angle of twist per unit length is given by

$$\frac{\theta}{L} = \frac{T}{k_2 d b^3 G} \qquad \text{...(2)}$$

k_1 and k_2 being two constants, their values depending on the ratio d/b and being given in Table 1.

d/b	1.0	1.5	1.75	2.0	2.5	3.0	4.0	6.0	8.0	10.0	∞
k_1	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
k_2	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Table 1: Table of k_1 and k_2 values for rectangular sections in torsion

Narrow rectangular sections

From Table 1 it is evident that as the ratio d/b increases, i.e. the rectangular section becomes longer and thinner, the values of constants k_1 and k_2 approach 0.333. Thus, for narrow rectangular sections in which d/b > 10 both k_1 and k_2 are assumed to be 1/3 and eqns. (1) and (2) reduce to

$$\tau_{max} = \frac{3T}{db^2} \qquad \text{...(3)}$$

$$\frac{\theta}{L} = \frac{3T}{db^3 G} \qquad \text{...(4)}$$

5.2. <u>Thin-walled open sections</u>

There are many cases, particularly in civil engineering applications, where rolled steel or extruded alloy sections are used where some element of torsion is involved. In most cases the sections consist of a combination of rectangles, and the relationships given in eqns. (1) and (2) can be adapted with reasonable accuracy provided that:

(a) the sections are "open", i.e. angles, channels, T-sections, etc., as shown in Fig. 1;

(b) the sections are thin compared with the other dimensions.

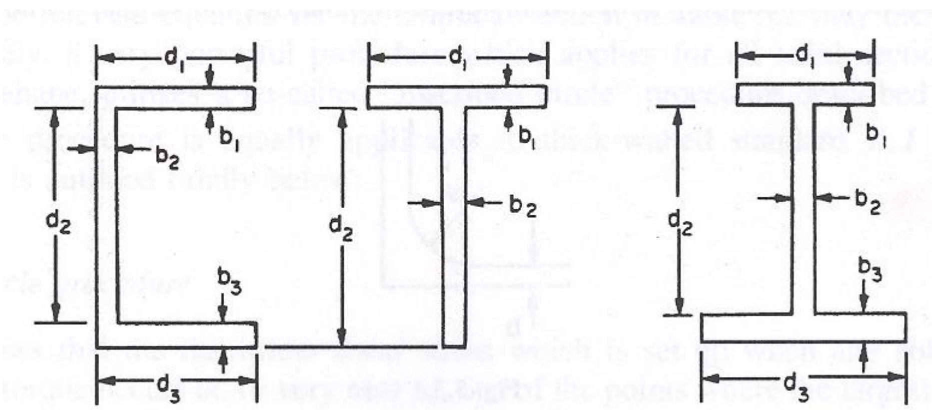


Fig. 1.Typical thin-walled open sections.

For such sections eqns. (1) and (2) may be re-written in the form

$$\tau_{max} = \frac{T}{\sum k_1 d b^2} \qquad \text{...(6)}$$

$$\frac{\theta}{L} = \frac{T}{G \sum k_2 d b^3} \qquad \text{...(5)}$$

and for d/b ratios in excess of 10, $k_1 = k_2 = 1/3$, so that

$$\tau_{max} = \frac{3T}{\sum d b^2}$$

$$\frac{\theta}{L} = \frac{3T}{G \sum d b^3}$$

5.3. <u>Thin-walled split tube</u>

The thin-walled split tube shown in Fig. 2 is considered to be a special case of the thin-walled open type of section considered in previous section. It is therefore treated as an equivalent rectangle with a longer side **d** equal to the circumference (less the gap), and a width **b** equal to the thickness.

Then $$\tau_{max} = \frac{T}{k_1 d b^2}$$

and $$\frac{\theta}{L} = \frac{T}{k_2 d b^3 G}$$

where k_1 and k_2 for thin-walled tubes are usually equal to 1/3.

It should be noted here that the presence of even a very small cut or gap in a thin-walled tube produces a torsional stiffness (torque per unit angle of twist) very much smaller than that for a complete tube of the same dimensions.

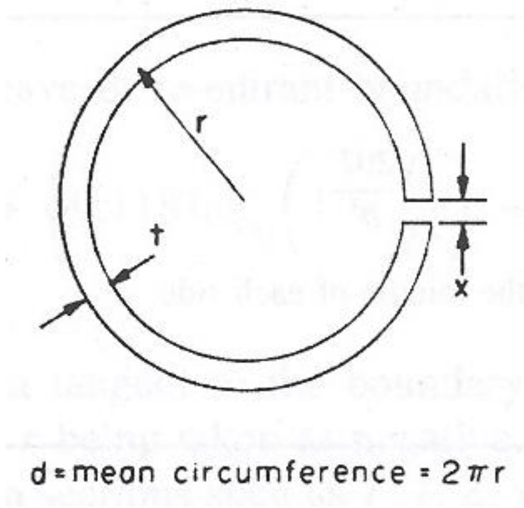


Fig. 2.Thin tube with longitudinal split.

5.4. <u>Other solid (non-tubular) shafts</u>

Table 2 indicates the relevant formulae for maximum shear stress and angle of twist of other standard non-circular sections which may be encountered in practice.

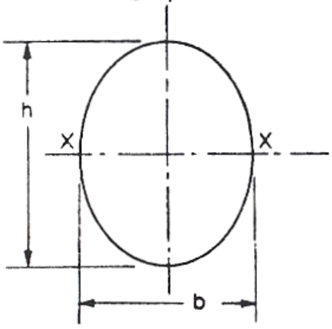
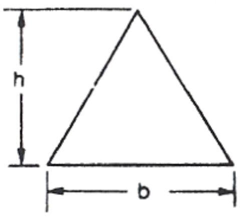
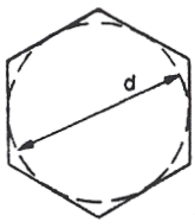
Approximate angles of twist for other solid cross-sections may be obtained by the substitution of an elliptical cross-section of the same area A and the same polar second moment of area J. The relevant equation for the elliptical section in Table 2 may then be applied.

Alternatively, a very powerful procedure which applies for all solid sections, however irregular in shape, utilizes a so-called “inscribed circle” procedure described in detail by Roark. The procedure is equally applicable to thick-walled standard T, I and channel sections and is outlined briefly below:

<u>Inscribed circle procedure</u>

Roark shows that the maximum shear stress which is set up when any solid section is subjected to torque occurs at, or very near to, one of the points where the largest circle whichcan be constructed within the cross-section touches the section boundary - see Fig. 3

Table 2: Relevant formulae for maximum shear stress and angle of twist

Cross-section	Maximum shear stress	Angle of twist per unit length
Elliptic 	$\frac{16T}{\pi b^2 h}$ at end of minor axis XX	$\frac{4\pi^2 TJ}{A^4 G}$
	where $J = \frac{\pi}{64}[bh^3 + hb^3]$ and A is the area of cross-section $= \pi bh/4$	
Equilateral triangle 	$\frac{20T}{b^3}$ at the middle of each side	$\frac{46.2T}{b^4 G}$
Regular hexagon 	$\frac{T}{0.217\ Ad}$	$\frac{T}{0.133\ Ad^2 G}$
	where d is the diameter of inscribed circle and A is the cross-sectional area	

Normally it occurs at the point where the curvature of the boundary is algebraically the least, convex curvatures being taken as positive and concave or re-entrant curvatures negative.
The maximum shear stress is then obtained from either:

$$\tau_{\max} = \left(\frac{G\theta}{L}\right) C \quad \text{or} \quad \tau_{\max} = \left(\frac{\tau}{K}\right) C$$

where, for positive curvatures (i.e. straight or convex boundaries),

$$C = \frac{D}{1 + \frac{\pi^2 D^4}{16A^2}}\left[1 + 0.15\left(\frac{\pi^2 D^4}{16A^2} - \frac{D}{2r}\right)\right]$$

with D = diameter of the largest inscribed circle,
r = radius of curvature of boundary at selected position (positive),
A = cross-sectional area of section,

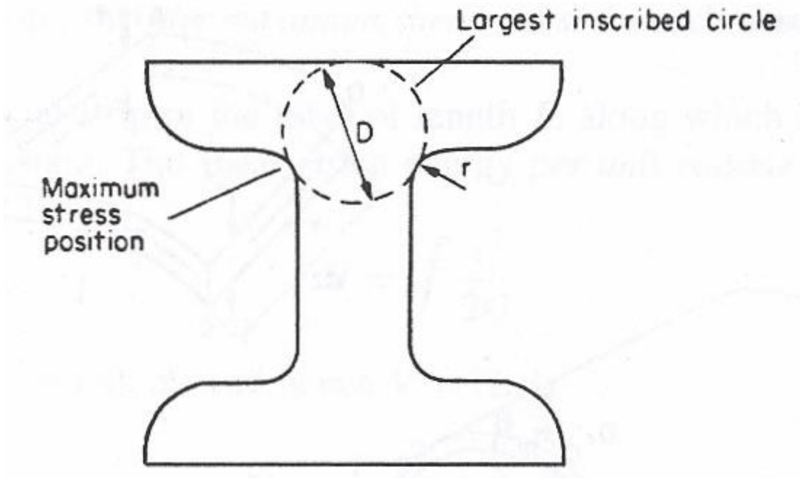


Fig. 3. Inscribed circle stress evaluation procedure.

or, for negative curvatures (concave or re-entrant boundaries):

$$C = \frac{D}{1 + \frac{\pi^2 D^4}{16A^2}} \left[1 + \left\{ 0.118 \log_e \left(1 - \frac{D}{2r} \right) - 0.238 \frac{D}{2r} \right\} \tanh \frac{2\phi}{\pi} \right]$$

with ϕ = angle through which a tangent to the boundary rotates in travelling around the re-entrant position (radians) and r being taken as negative.

For standard thick-walled open sections such as T, I, Z, angle and channel sections Roark also introduces formulae for angles of twist based upon the same inscribed circle procedure parameters.

5.5. <u>Thin-walled closed tubes of non-circular section</u>

Consider the thin-walled closed tube shown in Fig. 4 subjected to a torque T about the Z axis; i.e. in a transverse plane. Both the cross-section and the wall thickness around the periphery may be irregular as shown, but for the purposes of this simplified treatment it must be assumed that the thickness does not vary along the length of the tube. Then, if τ is the shear stress at B and τ' is the shear stress at C (where the thickness has increased to t') then, from the equilibrium of the complementary shears on the sides AB and CD of the element shown, it follows that

$$\tau t \, dz = \tau' t' \, dz$$

$$\tau t = \tau' t'$$

i.e. the product of the shear stress and the thickness is constant at all points on the periphery of the tube. This constant is termed the shear flow and denoted by the symbol q (shear force per unit length).

Thus $$q = \tau t = \textbf{constant}$$

The quantity q is termed the shear flow because if one imagines the inner and outer boundaries of the tube section to be those of a channel carrying a flow of water, then, provided that the total quantity of water in the system remains constant, the quantity flowing past any given point is also constant.

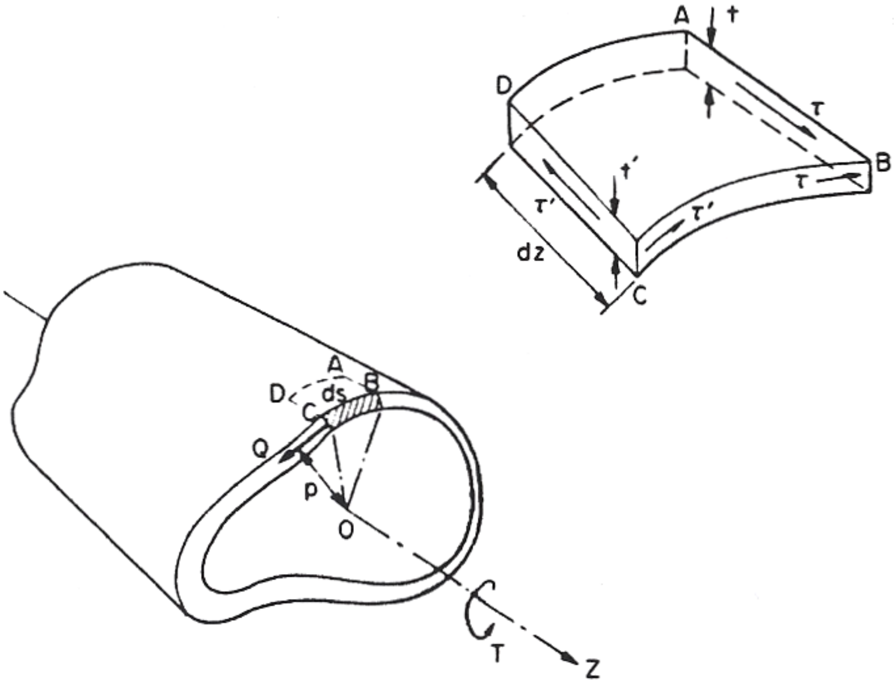


Fig.4. Thin-walled closed section subjected to axial torque.

At any point, then, the shear force Q on an element of length ds is Q = τ t ds = q ds and the shear stress is q/t.
Consider now, therefore, the element BC subjected to the shear force Q = qds = τ tds. The moment of this force about o

$$= dT = Qp$$

where p is the perpendicular distance from o to the force Q.

$$\therefore \qquad dT = q\,ds\,p$$

Therefore the moment, or torque, for the whole section

$$= \int qp\,ds = q \int p\,ds$$

But the area $COB = \frac{1}{2}$ base $\times$ height $= \frac{1}{2} p\,ds$

i.e. $$dA = \tfrac{1}{2} p\,ds \quad \text{or} \quad 2dA = p\,ds$$

$\therefore$ $$\text{torque } T = 2q \int dA$$

$$T = 2qA$$

where A is the area enclosed within the median line of the wall thickness.
Now, since

$$q = \tau t$$

it follows that $$T = 2\tau t A$$

or $$\tau = \frac{T}{2At}$$

where t is the thickness at the point in question.

It is evident, therefore, that the maximum shear stress in such cases occurs at the point of minimum thickness.
Consider now an axial strip of the tube, of length L, along which the thickness and hence the shear stress is constant. The shear strain energy per unit volume is given by

$$U = \int \frac{\tau^2}{2G}$$

Thus, with thickness t, width ds and hence V = tLds

$$U = \int \frac{\tau^2}{2G} tL\,ds$$

$$= \int \left(\frac{T}{2At}\right)^2 \frac{tL}{2G}\,ds$$

$$= \frac{T^2 L}{8A^2 G} \int \frac{ds}{t}$$

But the energy stored equals the work done $= \frac{1}{2} T\theta$

$\therefore$ $$\frac{1}{2} T\theta = \frac{T^2 L}{8A^2 G} \int \frac{ds}{t}$$

The angle of twist of the tube is therefore given by

$$\theta = \frac{TL}{4A^2 G} \int \frac{ds}{t}$$

For tubes of constant thickness this reduces to

$$\theta = \frac{TLs}{4A^2Gt} = \frac{\tau Ls}{2AG}$$

where s is the perimeter of the median line.
The above equations must be used with care and do not apply to cases where there are abrupt changes in thickness or re-entrant corners.
For closed sections which have constant thickness over specified lengths but varying from one part of the perimeter to another:

$$\frac{\theta}{L} = \frac{T}{4A^2G}\left[\frac{s_1}{t_1} + \frac{s_2}{t_2} + \frac{s_3}{t_3} + \cdots \text{etc.}\right]$$

EXAMPLES

1.A rectangular steel bar 25 mm wide and 38 mm deep is subjected to a torque of 450 Nm. Estimate the maximum shear stress set up in the material of the bar and the angle of twist, using the experimentally derived formulae stated in §1.

What percentage error would be involved in each case if the approximate equations areused? For steel, take G = 80 GN/m^2.

Solution

The maximum shear stress is given by eqn. (1):

$$\tau_{max} = \frac{T}{k_1 d b^2}$$

In this cased = 38mm, b = 25 mm, i.e. d/b = 1.52 and k1 for d/b of 1.5 = 0.231.

$$\therefore \qquad \tau_{max} = \frac{450}{0.231 \times 38 \times 10^{-3} \times (25 \times 10^{-3})^2} = \mathbf{82\ MN/m^2}$$

The angle of twist per unit length is given by eqn. (2):

$$\frac{\theta}{L} = \frac{T}{k_2 d b^3 G}$$

and from the tables, for d/b = 1.5, k_2 is 0.196.

$$\therefore \qquad \theta = \frac{450}{0.196 \times 38 \times 10^{-3} \times (25 \times 10^{-3})^3 \times 80 \times 10^9}$$
$$= 0.0483 \text{ rad/m}$$
$$= \mathbf{2.77\ degrees/m}$$

Approximately

$$\tau_{max} = \frac{T}{db^2}(3 + 1.8b/d)$$
$$= \frac{450}{38 \times 10^{-3} \times (25 \times 10^{-3})^2}\left(3 + 1.8 \times \frac{25}{38}\right)$$
$$= \frac{450}{2.375 \times 10^{-5}}(3 + 1.184) = \mathbf{79.3\ MN/m^2}$$

Therefore percentage error

$$= \left(\frac{79.3 - 82.02}{82.02}\right) 100 = \mathbf{-3.3\%}$$

Again approximately,

$$\theta = \frac{42TJ}{GA^4} \text{ per metre}$$

$$\text{Now} \qquad J = I_{xx} + I_{yy} = \frac{bd^3}{12} + \frac{db^3}{12} = \frac{bd}{12}(d^2 + b^2)$$

$$= \frac{25 \times 38(25^2 + 38^2)}{12 \times 10^{12}} = 0.1638 \times 10^{-6}\ \text{m}^4$$

$$\therefore \qquad \theta = \frac{42 \times 450 \times 0.164 \times 10^{-6}}{80 \times 10^9 \times (25 \times 38 \times 10^{-6})^4} = 0.0476\ \text{rad/m}$$

$$= 2.73\ \text{degrees/m}$$

$$\text{Percentage error} = \left(\frac{2.73 - 2.77}{2.77}\right) 100 = \mathbf{-1.44\%}$$

2. Compare the torsional stiffness of the following cross-sections which can be assumed to be of unit length. Compare also the maximum shear stresses set up in each case:

(a) a hollow tube 40 mm mean diameter and 2 mm wall thickness;

(b) the same tube with a 2 mm wide saw-cut along its length;

(c) a rectangular solid bar, side ratio 4 to 1, having the same cross-sectional area as that enclosed by the mean diameter of the hollow tube;

(d) an equal-leg angle section having the same perimeter and thickness as the tube;

(e) a square box section having the same perimeter and thickness as the tube.

Solution

(a) In the case of the closed hollow tube we can apply the standard torsion equation

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}$$

together with the simplified formula for the polar moment of area J of thin tubes,

$$J = 2\pi r^3 t$$

$$\therefore \qquad \text{torsional stiffness} = \frac{T}{\theta} = \frac{GJ}{L} = \frac{2\pi \times (20 \times 10^{-3})^3 \times 2 \times 10^{-3} G}{1}$$

$$= 100.5 \times 10^{-9} G$$

$$\text{maximum shear stress} = \frac{TR}{J} = \frac{20 \times 10^{-3} \times T}{2\pi \times (20 \times 10^{-3})^3 \times 2 \times 10^{-3}}$$

$$= \mathbf{0.198 \times 10^6 T}$$

(b) Tube with split From the work of §4,

$$\text{angle of twist/unit length} = \frac{\theta}{L} = \frac{T}{k_2\, db^3 G} = \frac{T}{k_2(2\pi r - x)t^3 G}$$

$$\therefore \qquad \text{torsional stiffness} = \frac{T}{\theta} = \frac{k_2(2\pi r - x)t^3 G}{L}$$

$$= \frac{0.333[2\pi \times 20 \times 10^{-3} - 2 \times 10^{-3}](2 \times 10^{-3})^3 G}{1}$$

$$= 0.333(125.8 - 2)8 \times 10^{-12} G$$

$$= \mathbf{329.8 \times 10^{-12} G}$$

$$\text{Maximum shear stress} = \frac{T}{k_1 d b^2}$$

$$= \frac{T}{0.333 \times 123.8 \times 10^{-3} \times (2 \times 10^{-8})^2}$$

$$= \mathbf{6.06 \times 10^6 T}$$

i.e. splitting the tube along its length has reduced the stiffness by a factor of approximately 300, the maximum stress increasing by approximately 30 times.

(c) Rectangular bar

Area of hollow tube = area of bar

$$= \pi \times (20 \times 10^{-3})^2$$

$$\therefore \qquad 4b^2 = 8\pi \times 10^{-4}$$

$$b^2 = 2\pi \times 10^{-4}$$

$$\therefore \qquad b = 2.5 \times 10^{-2} \text{ m} = 25 \text{ mm}$$

$$\therefore \qquad d = 4b = 100 \text{ mm}$$

d/b ratio $= 4$

$$\therefore \qquad k_1 = 0.282 \quad \text{and} \quad k_2 = 0.281$$

Therefore from eqn. (5.2),

$$\frac{\theta}{L} = \frac{T}{k_2 d b^3 G}$$

$$\therefore \qquad \frac{T}{\theta} = 0.281 \times 10 \times 10^{-2} \times (2.5 \times 10^{-2})^3 G$$

$$= 43.9 \times 10^{-8} G$$

$$= \mathbf{439 \times 10^{-9} G}$$

From eqn. (1),

$$\tau_{max} = \frac{T}{k_1 d b^2} = \frac{T}{0.282 \times 10 \times 10^{-2} \times (2.5 \times 10^{-2})^2}$$

$$= \mathbf{0.057 \times 10^6 T}$$

(d) Equal-leg angle section

Perimeter of angle = perimeter of tube

= 2p x 20 x 10^{-3} m

Length of side d = 20p x 10^{-3}

Therefore applying the following equation,

$$\frac{\theta}{L} = \frac{3T}{G\Sigma db^3}$$

$$= \frac{3T}{2G \times 20\pi \times 10^{-3} \times (2 \times 10^{-3})^3}$$

$$\therefore \qquad \frac{T}{\theta} = (2G \times 20\pi \times 8 \times 10^{-12})/3$$

$$= \mathbf{0.335 \times 10^{-9}G}$$

And from eqn. below:

$$\tau_{max} = \frac{3T}{\Sigma db^2}$$

$$= \frac{3T}{2 \times 20\pi \times 10^{-3} \times (2 \times 10^{-3})^2}$$

$$= \mathbf{5.97 \times 10^6\ T}$$

(e) Square box section (closed)

Perimeters = tube perimeter = 2p x 20 x I 0^{-3}m

$$\therefore \qquad \text{side length} = \frac{2\pi \times 20 \times 10^{-3}}{4} = \pi \times 10^{-2}\ \text{m}$$

Therefore area enclosed by median line

$$A = (\pi \times 10^{-2})^2$$

From eqn. below

$$\theta = \frac{TLs}{4A^2Gt}$$

$$\therefore \qquad \frac{T}{\theta} = \frac{4 \times (\pi \times 10^{-2})^4 G \times 2 \times 10^{-3}}{1 \times 2\pi \times 20 \times 10^{-3}}$$

$$= \mathbf{62 \times 10^{-9}G}$$

From eqn. below

$$\tau_{max} = \frac{T}{2At} = \frac{T}{2 \times (\pi \times 10^{-2})^2 \times 2 \times 10^{-3}}$$

$$= \mathbf{0.253 \times 10^6 T}$$

3.A thin-walled member 1.2 m long has the cross-section shown in Fig.1. Determine the maximum torque which can be carried by the section if the angle of twist is limited to 100. What will be the maximum shear stress when this maximum torque is applied? For the material of the

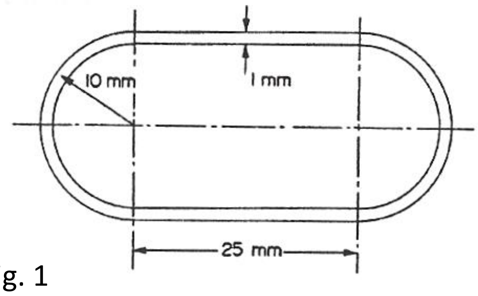


Fig. 1

member G = 80 GN/m^2.

Solution

This problem is depending upon the length, and the area enclosed by, the median line.

Now, perimeter of median line $= s = (2 \times 25 + 2\pi \times 10)$ mm

$$= 112.8 \text{ mm}$$

area enclosed by median $= A = (20 \times 25 + \pi \times 10^2)\ \text{mm}^2$

$$= 814.2 \text{ mm}^2$$

Using equation below:

$$\theta = \frac{TLs}{4A^2\,Gt}$$

$$\therefore \quad \frac{10 \times 2\pi}{360} = \frac{T \times 1.2 \times 112.8 \times 10^{-3}}{4(814.2 \times 10^{-6})^2 \times 80 \times 10^9 \times 1 \times 10^{-3}}$$

i.e. maximum torque possible,

$$T = \frac{20\pi \times 4 \times 814.2^2 \times 80 \times 10^{-6}}{360 \times 1.2 \times 112.8 \times 10^{-3}}$$

$$= \mathbf{273\ Nm}$$

Using equation below

$$\tau_{max} = \frac{T}{2At}$$

$$= \frac{273}{2 \times 814.2 \times 10^{-6} \times 1 \times 10^{-3}}$$

$$= 168 \times 10^6 = \mathbf{168\ MN/m^2}$$

The maximum stress produced is 168 MN/m^2.